On calculation of effective galvanomagnetic characteristics of inhomogeneous metals. Exact solution for the longitudinal effective conductivity of polycrystals of metals in high magnetic fields.

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Abstract

In the framework of the perturbation theory an expression suitable for calculation of the effective conductivity of 3-D inhomogeneous metals in uniform magnetic field H is derived. For polycrystals of metals with closed Fermi surfaces in high magnetic fields the perturbation series defining the longitudinal and the hall elements of the perturbation series can be summed allowing us to obtain the exact expression for the leading terms of all these elements of the effective conductivity tensor.

1   Introduction

Calculation of the effective (macroscopic) conductivity of inhomogeneous media (in particular, of a randomly inhomogeneous medium) is one of the well-known problems drawing attention of scientists during a very long time. However, in the general case this problem has not been solved yet. In this Introduction we do not pretend to give the full list of references, but refer the readers to the review paper by A.G.Fokin [1], where the long list of references relating to different aspects of the problem is presented. We concentrate mainly on the known exact solutions for the effective conductivity tensor (ECT).

Exact solutions for effective characteristics of stochastically inhomogenous media can be found very rarely. With regard to calculation of ECT the simplest exact solution was obtained by Wiener as far as 1912 (see Ref. [1]). He examined an unbounded plane-parallel layered medium, where the conductive properties of a layer are described by the conductivity tensor for one of N homogeneous isotropic components.

The other example is the Dykhne formula for ECT of some of two- dimensional inhomogeneous media [2]. Also for a plasma located in a strong magnetic field, in [3] A.M.Dykhne obtained exact expressions for the effective conductivity and the effective Hall parameter, supposing that the density fluctuations are two-dimensional ones. The existence of these results is due to specific transformations allowed by the equations of 2-D problems. Recently S.A.Bulgadaev [4] generalized the Dykhne formula obtaining new exact solutions for 2-D heterophase medium with an arbitrary number of phases N.

Some time ago A.M.Dykhne and I.M.Kaganova [5, 6] derived the exact solution for the effective surface impedance of an inhomogeneous metal in the frequency region of the local impedance (the Leontovich) boundary condition applicability. In this case the existence of the exact solution is caused by the presence of the physically meaningful small parameter, namely, the surface impedance of good metals. The result is valid both

under conditions of normal and anomalous skin effect, allowing us to calculate effective impedance of strongly anisotropic polycrystalline metals [6, 7]. The inhomogeneity of the metallic surface can be due to the properties of the metal or/and surface roughness. The effective impedance of strongly curved one-dimensional metallic surface was discussed in [8].

Sometimes effective characteristics can be estimated qualitatively. For example, there are no regular methods allowing us to calculate the effective conductivity of 3-D polycrystals accurately, but some general reasoning suggested by Yu.A.Dreizin and A.M.Dykhne [9] allowed them to estimate ECT of strongly anisotropic polycrystals.

When the exact solution for an effective characteristic cannot be found, perturbation theory calculations may be useful. The most accurate and physically meaningful method to take account of spatial fluctuations of the characteristic was developed by I.M.Lifshits and his co-workers. (The first paper related to this topic was [10]). They proposed to start from equations of the problem coefficients of which are random functions of position. Averaging these equations one derives equations for the averaged fields, which allow him to determine the effective characteristic. This method is relatively simple, when the spatial fluctuations are small. Mostly it is used when only the first nonvanishing term of the perturbation series is taken into account. Usually, this term does not depend on the correlations of the stochastic characteristic in different points of the medium.

In [11] in the absence of a uniform magnetic field the perturbation theory formula, suitable when calculating the effective conductivity (EC) of 3-D inhomogeneous metals was derived. When the perturbation series converges, the result of [11] allows one to calculate EC as the series in powers of invariants of the local conductivity tensor (LCT) $\sigma_{ik}(\mathbf{r})$. EC of 3-D polycrystals was examined. The first term of the series taking account of spatial fluctuations of LCT is of the second order. As usual, if the polycystal is isotropic in average, this term does not depend on statistical properties of the inhomogeneous medium. However, beginning from the third order term the answer depends on the correlations of the fluctuating part of LCT in different points of the medium.

In this paper we present the perturbation theory formula for the effective conductivity tensor (ECT) in a uniform magnetic field $\mathbf{H}$. We examine stochastically inhomogeneous metals that are isotropic and homogeneous in average. The way of deriving the expression for ECT is much similar to the algorithm proposed in [11]. We apply the obtained result when calculating ECT of polycrystalline metals, composed of single crystal grains, supposing that the Fermi surface of the single crystal is a closed one. The case of very high magnetic fields ($r_c \ll l$, where r_c is the cyclotron radius and l is the electron mean free path) is examined. We show that under the abovementioned conditions the perturbation series relating to the longitudinal element and the Hall element of ECT can be summed, allowing us to calculate the leading terms of these elements of ECT exactly. The existence of this exact solution is due to the specific form of LCT when $r_c/l \ll 1$.

Classical galvanomagnetic properties of single crystal metals were scrutinized by a lot of authors. The results can be found in textbooks of electron theory of metals (see, for example, [12]). In the general case the dependence of the elements of the single crystal conductivity tensor on the value and direction of the magnetic field cannot be obtained for an arbitrary value of H. The elements of the conductivity tensor depend on the details of the scattering processes, as well as on the dynamics of conduction electrons. However, the asymptotic behavior of the elements of the conductivity tensor in high magnetic fields is defined mainly by the structure of the Fermi surface of the metal.

The Fermi surfaces of real metals are extremely complex and differ significantly for different metals[13]. The first paper where the magnetoresistivity of polycrystals of metals was calculated with regard to the presence of open electron orbits, was the work by I.M.Lifshitz and V.G.Peschanskii [14]. In [15] Yu.A.Dreizin and A.M.Dykhne discussed ECT of polycrystals of metals with open Fermi surfaces (of the "space mesh" and "corrugated cylinder" types). In this case in some specifically oriented grains conduction electrons move along open orbits. Due to the contribution of these grains to the transverse effective conductivity, its leading term being nontrivially dependent on H, is anomalously large.

However, up to now ECT for polycrystals of metals with closed Fermi surfaces has not been examined. Our perturbation theory allows us to examine this type of polycrystalline metals. Note, even the shape of a closed Fermi surface can be extremely complex. We restrict ourselves with the case when the number of conduction electrons n_e is not equal to the number of holes n_h. The case $n_e = n_h$, as well as weak uniform magnetic fields will be discussed elsewhere.

The organization of this paper is as follows. In Section 2 with the aid of perturbation theory we derive an expression allowing us to calculate ECT of 3-D inhomogeneous metals in a uniform magnetic field. In Section 3 we use the obtained result to examine ECT of polycrystals of metals with closed Fermi surfaces in high uniform magnetic fields. The obtained results are exact, when the single crystal anisotropy is not very strong and perturbation series converges. As an example, we calculate the effective longitudinal conductivity for polycrystals of metals with an ellipsoidal uniaxial Femi surface. Concluding remarks are given in Section 4.

2 Perturbation Theory for ECT of 3-D Inhomogeneous Metals in Uniform Magnetic Field

If $\mathbf{H} \neq 0$, the elements of LCT $\sigma_{ik}(\mathbf{H}, \mathbf{r})$ are random functions of position vector $\mathbf{r}$, and in the same time they depend on the value and orientation of the fixed vector $\mathbf{H}$. By $< ... >$ denote the ensemble average over all possible realizations of the medium. When $\mathbf{H} = 0$, the averaged conductivity $< \sigma_{ik} >=< \sigma > \delta_{ik}$, however, if $\mathbf{H} \neq 0$, the averaged conductivity $< \sigma_{ik}(\mathbf{H}) >$ is a tensor, which does not reduce to the isotropic tensor δ_{ik} only. Let us write LCT as

$$\sigma_{ik}(\mathbf{H}, \mathbf{r}) =< \sigma_{ik}(\mathbf{H}) > +\Delta_{ik}(\mathbf{H}, \mathbf{r}); \quad < \Delta_{ik}(\mathbf{H}, \mathbf{r}) >= 0, \tag{1}$$

where $\Delta_{ik}(\mathbf{H}, \mathbf{r})$ are random functions of $\mathbf{r}$. The tensor $\Delta_{ik}(\mathbf{H}, \mathbf{r})$ is responsible for the spatial fluctuations of LCT.

By definition, ECT σ_{ik}^{ef} is specified by equation

$$< j_i >= \sigma_{ik}^{ef} < E_k >, \tag{2}$$

where $< \mathbf{j} >$ is the macroscopic direct current density and $< \mathbf{E} >$ is the uniform macroscopic electric field. Let $\mathbf{j}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ be the local current density and the local electric field, respectively. They are solutions of the electrostatics equations

$$\mathrm{div}\mathbf{j} = 0, \quad \mathrm{rot}\mathbf{E} = 0, \tag{3}$$

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with the material equation that is the Ohm law:

$$j_i(\mathbf{r}) = (<\sigma_{ik}(\mathbf{H})> +\Delta_{ik}(\mathbf{H},\mathbf{r}))E_k(\mathbf{r}). \tag{4}$$

Because of the tensor $\Delta_{ik}(\mathbf{H},\mathbf{r})$, Eq.(3) constitute a system of stochastic equations for the local field $E_i(\mathbf{r})$.

We set $\mathbf{j}(\mathbf{r}) =<\mathbf{j}> +\delta\mathbf{j}$ and $\mathbf{E}(\mathbf{r}) =<\mathbf{E}> +\delta\mathbf{E}$, where $<\delta\mathbf{j}>=<\delta\mathbf{E}>= 0$. Substituting these expressions in Eq.(4) and averaging, we obtain

$$<j_i>=<\sigma_{ik}(\mathbf{H})><E_k> +J_i; \quad J_i =<\Delta_{ik}(\mathbf{H},\mathbf{r})\delta E_k>. \tag{5}$$

The uniform vector $\mathbf{J}(\mathbf{H})$ describes the contribution of the spatial fluctuations of LCT to the elements of ECT.

Subtracting Eq.(5) from Eq.(4) we have

$$\delta j_i =<\sigma_{ik}(\mathbf{H})> \delta E_i+\Delta_{ik}(\mathbf{H},\mathbf{r})<E_k> +D_i(\mathbf{H},\mathbf{r}), \quad D_i(\mathbf{H},\mathbf{r}) = \Delta_{ik}(\mathbf{H},\mathbf{r})\delta E_k - J_i(\mathbf{H}). \tag{6}$$

Evidently, $<\mathbf{D}>= \mathbf{0}$. The components of the vector $\mathbf{D}$ are at least quadratic in powers of the elements of the tensor Δ_{ik}. When calculating ECT up to the first nonvanishing term taking account of spatial fluctuations, the vector $\mathbf{D}$ has to be omitted. However, just this vector defines corrections of higher orders.

From the electrostatic equations (3) it follows that the Fourier coefficient of the stochastic electric field $\delta E_i(\mathbf{r})$ is

$$\delta E_i(\mathbf{k}) = -q_{ik}(\Delta_{jk}(\mathbf{H},\mathbf{k})<E_k> +D_j(\mathbf{H},\mathbf{k})), \tag{7.a}$$

where $\Delta_{jk}(\mathbf{H},\mathbf{k})$ and $D_j(\mathbf{H},\mathbf{k})$ are the Fourier coefficients of the elements of the stochastic tensor $\Delta_{jk}(\mathbf{H},\mathbf{r})$ and the components of the stochastic vector $D_i(\mathbf{H},\mathbf{r})$, respectively, and the tensor q_{ik} is

$$q_{ik} = \frac{k_i k_k}{k^2 Q}, \quad Q =<\sigma_{ik}(\mathbf{H})> \frac{k_i k_k}{k^2}, \tag{7.b}$$

We seek δE_i and J_i as series in powers of the elements of the tensor Δ_{ik}:

$$\delta E_i(\mathbf{r}) = \sum_{n=1}^{\infty} \delta E_i^{(n)}(\mathbf{r}); \quad J_i = \sum_{n=2}^{\infty} J_i^{(n)}. \tag{8}$$

With respect to the definition of $\mathbf{J}$, we have $J_i^{(n)} =<\Delta_{jk}(\mathbf{H},\mathbf{r})\delta E_i^{(n-1)}>$. Next, from Eq.(7.a) it follows that the Fourier coefficients of $\delta E_i^{(n)}(\mathbf{r})$ are

$$\delta E_i^{(1)}(\mathbf{k}) = -q_{ij}\Delta_{jk}(\mathbf{H},\mathbf{k})<E_k> \quad \text{and} \quad \delta E_i^{(n)}(\mathbf{k})|_{n>1} = -q_{ij}D_j^{(n)}(\mathbf{H},\mathbf{k}), \tag{9}$$

where $D_j^{(n)}(\mathbf{H},\mathbf{k})$ is the Fourier coefficient of $D_j^{(n)}(\mathbf{r}) = \Delta_{jk}(\mathbf{H},\mathbf{r})\delta E_k^{(n-1)}(\mathbf{r}) - J_j^{(n)}(\mathbf{H})$.

We use Eq.(9) to write $\delta E_i^{(n)}(\mathbf{r})$ as a sum of the lower-order terms. Indeed, suppose $n \geq 2$. Then with regard to the definition of the vector $D_j^{(n)}(\mathbf{r})$ we have

$$\delta E_i^{(n)}(\mathbf{r}) = -\left\{ \int\int d^3 k_n d^3 k_{n-1} q_{ij_n}^{(n)}(\mathbf{H})\Delta_{j_n k_{n-1}}(\mathbf{H};\mathbf{k}_n - \mathbf{k}_{n-1})\delta E_{k_{n-1}}^{(n-1)}(\mathbf{k}_{n-1})e^{i\mathbf{k}_n\mathbf{r}} - I_{ik}(\mathbf{H})J_k^{(n)} \right\}. \tag{10.a}$$

where $q_{ik}^{(n)}$ is the tensor q_{ik} for $\mathbf{k} = \mathbf{k}_n$, and

$$I_{ik}(\mathbf{H}) = \int \mathrm{d}^3 k q_{ik}(\mathbf{H})\delta(\mathbf{k}), \tag{10.b}$$

$\delta(\mathbf{k})$ is the δ-function.

We use Eq.(10.a) to present $\delta E_{k_{n-1}}^{(n-1)}(\mathbf{r})$ as a functional of $\delta E_j^{(n-2)}$. After Fourier analyzing the obtained expression, we substitute it in Eq.(10.a). If in the subsequent order we repeat this decreasing procedure, we obtain

$$\sum_{l=2}^{m} v_{ik}^{(m-l)} J_k^{(l)} = -w_{ik}^{(m)} < E_k >, \tag{11}$$

where

$$v_{ik}^{(0)} = \delta_{ik}, \quad v_{ik}^{(1)} = 0, \quad v_{ik}^{(n)} = w_{im}^{(n)} I_{mk}, \text{ if } n \geq 2. \tag{12}$$

With the replacement of $\Delta_{j_r l_{r+1}}(\mathbf{H}, \mathbf{k}_r - \mathbf{k}_{r+1})$ by $\Delta_{j_r l_{r+1}}(\mathbf{H}, \mathbf{r}_r)$ in the integrand of the expression for $w_{ik}^{(n)}$, we obtain

$$w_{ik}^{(n)}(\mathbf{H}) = \frac{(-1)^n}{(2\pi)^{3(n-1)}} \int ... \int \mathrm{d}^3 k_1 ... \mathrm{d}^3 k_{n-1} e^{i\mathbf{k}_1 \mathbf{r}} q_{l_1 j_1}^{(1)} q_{l_2 j_2}^{(2)} ... q_{l_{n-1} j_{n-1}}^{(n-1)} \times$$

$$\int ... \int \mathrm{d}^3 r_1 ... \mathrm{d}^3 r_{n-1} e^{-i\mathbf{r}_1(\mathbf{k}_1 - \mathbf{k}_2)} e^{-i\mathbf{r}_2(\mathbf{k}_2 - \mathbf{k}_3)} ... e^{-i\mathbf{r}_{n-1}\mathbf{k}_{n-1}} \times$$

$$< \Delta_{il_1}(\mathbf{H}, \mathbf{r})\Delta_{j_1 l_2}(\mathbf{H}, \mathbf{r}_1) ... \Delta_{j_{n-1} k}(\mathbf{H}, \mathbf{r}_{n-1}) > . \tag{13}$$

If we introduce

$$W_{ik} = \sum_{m=2}^{\infty} w_{ik}^{(m)}; \quad V_{ik} = \sum_{m=0}^{\infty} v_{ik}^{(m)} = \delta_{ik} + W_{im} I_{mk}, \tag{14}$$

summing the equations (11) for all $2 \leq m < \infty$, we have

$$J_i = -V_{ij}^{-1} W_{jk} < E_k > . \tag{15}$$

Consequently, according to Eq.(5)

$$\sigma_{ik}^{ef} = < \sigma_{ik}(\mathbf{H}) > -V_{ij}^{-1}(\mathbf{H})W_{jk}(\mathbf{H}). \tag{16}$$

If for a given uniform magnetic field we know LCT and the inhomogeneity of the metal is not very strong (the perturbation series converge), Eq.(16) allows us to calculate ECT of an inhomogeneous metal with an arbitrary accuracy with respect to inhomogeneity. However, to calculate the many-point correlators in the integrand of Eq.(13), we need to define certain statistical properties of our inhomogeneous medium. Usually, the dependence on the statistical properties manifests itself, when after calculation of the contractions of mani-point averages with the tensors q_{ik} (see Eq.(7.b)) in the right-hand side of Eq.(13), scalar products $\mathbf{k}_p \mathbf{k}_q$ ($p \neq q$) enter the integrand.

In the next Section we show that sometimes even in the 3-D case summing the series (14), we can calculate some elements of ECT exactly. For example, this is true for the longitudinal conductivity of polycrystals of metals with closed Fermi surfaces in high magnetic fields.

3 Exact solution for the effective conductivity of polycrystals in high magnetic fields

Polycrystals are widespread case of inhomogeneous media, where the inhomogeneity is due to different orientation of discrete single crystal grains. If crystallographic axes of the grains are randomly rotated with respect to a fixed set of laboratory axes, the elements of LCT measured in the laboratory coordinate system are stochastic functions of position. When calculating the conductivity tensor in a uniform magnetic field, it is natural to connect the laboratory axes with the direction of the vector $\mathbf{H}$. In what follows we assume that the magnetic vector is directed along the axis 3 of the laboratory coordinate system.

We examine polycrystals of metals with closed Fermi surfaces supposing that number of conduction electrons n_e is not equal to the number of holes n_h. We examine the case of very high magnetic fields ($r_c \ll l$). Under these suppositions the invariant form of the asymptotic expression for LCT $\sigma_{ik}(\mathbf{H}, \mathbf{r})$ can be written as (see, e.g., [12])

$$\sigma_{ik}(\mathbf{H}, \mathbf{r})|_{H \to \infty} = S(\vec{\kappa}, \vec{r})\kappa_i \kappa_k + \frac{1}{H} e_{ikl} a_{lm}(\vec{\kappa}, \vec{r})\kappa_m + \frac{1}{H^2} s_{ik}(\vec{\kappa}, \vec{r}), \qquad (17)$$

where $\kappa_i = H_i/H$ (in the laboratory coordinate system $\vec{\kappa} = (0, 0, 1)$) and $s_{ik} = s_{ki}$. We also suppose that due to the symmetry of our single crystal $a_{ik} = a_{ki}$. The elements of the tensors s_{ik}, a_{ik} as well as the scalar S depend on the fixed vector $\vec{\kappa} = \vec{H}/H$. The expression (17) is the expansion of of the elements of the tensor $\sigma_{ik}(\mathbf{H}, \mathbf{r})$ in powers of the small parameter r_c/l: the first term corresponds to the leading term of the longitudinal local conductivity that is independent of the parameter $1/H$, the second term is the Hall conductivity, and the term proportional to $1/H^2$ contributes to the transverse conductivity.

Our goal is to calculate the leading terms of the elements of ECT. This means that we calculate the longitudinal effective conductivity up to the terms independent of the parameter $1/H$. We will show that fluctuating part of LCT does not enter the effective Hall conductivity. Also our calculations show that the transverse effective conductivity cannot be calculated with the aid of the approach used when calculating the longitudinal and the Hall effective conductivites. The difficulties one faced when calculating the transverse effective conductivity are clearly seen from the results of [15].

For our calculations it is important that in the laboratory coordinate system the local element $\sigma_{12}^{(H)}$ of the Hall conductivity is defined by the difference $n_e - n_h$ only: $\sigma_{12}^{(H)} = 1/R_\infty H$, where $R_\infty = 1/ec(n_e - n_h)$ is the Hall constant. In the invariant notation $\sigma_{12}^{(H)} = a_{ik}\kappa_i\kappa_k$. Thus, for all the grains the contraction $a_{ik}(\mathbf{H}, \mathbf{r})\kappa_i\kappa_k = \langle a_{ik}\kappa_i\kappa_k \rangle = 1/R_\infty$.

The ensemble averages $\langle s_{ik} \rangle$ and $\langle a_{ik} \rangle$ are symmetric isotropic second rank tensors whose elements depend on $\vec{\kappa}$. The general form of a symmetric isotropic tensor $M_{ik}(\vec{\kappa})$ is

$$M_{ik}(\vec{\kappa}) = M_1 \delta_{ik} + M_2 \kappa_i \kappa_k,$$

$$M_1 = \frac{1}{2}[M_{kk} - M_{ik}\kappa_i\kappa_k], \quad M_2 = \frac{1}{2}[3M_{ik}\kappa_i\kappa_k - M_{kk}]. \qquad (18)$$

Consequently, if we set $\langle s_{ik} \rangle = S_1 \delta_{ik} + S_2 \kappa_i \kappa_k$ and $\langle a_{ik} \rangle = A_1 \delta_{ik} + A_2 \kappa_i \kappa_k$, we have

$$S_1 = \frac{1}{2}[\langle s_{kk} \rangle - \langle s_{ik} \rangle \kappa_i \kappa_k], \quad S_2 = \frac{1}{2}[3\langle s_{ik} \rangle \kappa_i \kappa_k - \langle s_{kk} \rangle]. \qquad (19.a)$$

and
$$A_1 = \frac{1}{2}[<a_{kk}> -1/R_\infty], \quad A_2 = \frac{1}{2}[3/R_\infty - <a_{kk}>]. \tag{19.b}$$

When writing Eq.(19.b) we used the equality $<a_{ik}\kappa_i\kappa_k> = 1/R_\infty$.

Thus, averaging LCT (17) we obtain

$$<\sigma_{ik}(\mathbf{H})>|_{H\to\infty} = <S(\vec{\kappa})>\kappa_i\kappa_k + \frac{1}{HR_\infty}e_{ikm}\kappa_m + \frac{1}{H^2}[S_1\delta_{ik} + S_2\kappa_i\kappa_k]. \tag{20}$$

If $s_{ik} = <s_{ik}> +\delta s_{ik}$, $a_{ik} = <a_{ik}> +\delta a_{ik}$ and $S = <S> +\delta S$ ($<\delta s_{ik}> = <\delta s_{ik}> = <\delta S> = 0$), the fluctuating part of the LCT is

$$\Delta_{ik}(\mathbf{H}, \mathbf{r})|_{H\to\infty} = \delta S(\vec{\kappa}, \vec{r})\kappa_i\kappa_k + \frac{1}{H}e_{ikl}\delta a_{lm}(\vec{\kappa}, \vec{r})\kappa_m + \frac{1}{H^2}\delta s_{ik}(\vec{\kappa}, \vec{r}). \tag{21.a}$$

With regard to the abovementioned argumentation, the contraction

$$\delta a_{ik}\kappa_i\kappa_k = 0. \tag{21.b}$$

To make use of Eq.(16), we first calculate the terms $w_{ik}^{(n)}$ of the series W_{ik} (see Eq.(15)). Let us write down the product of the tensors $\Delta_{il_1}(\mathbf{H}, \mathbf{r})\Delta_{j_1l_2}(\mathbf{H}, \mathbf{r}_1)...\Delta_{j_{n-1}k}(\mathbf{H}, \mathbf{r}_{n-1})$ entering the integrand of the right-hand side of Eq.(13). Having in mind to calculate the effective longitudinal and the Hall conductivities only, we write this product up to the terms of the order of $1/H$. Then

$$\Delta_{il_1}(\mathbf{H}, \mathbf{r})\Delta_{j_1l_2}(\mathbf{H}, \mathbf{r}_1)...\Delta_{j_{n-1}k}(\mathbf{H}, \mathbf{r}_{n-1}) = (\hat{L}^{(n)} + \frac{1}{H}\hat{A}^{(n)} + ...)\hat{P}^{(n)}, \tag{22}$$

where the hat above the letter denotes a set of subscripts. In this notations $\hat{P}^{(n)}$ is the product

$$\hat{P}^{(n)} = \prod_{q=1}^{n-1} \kappa_{l_q}\kappa_{j_q}. \tag{23}$$

In Eq.(22) the tensor $\hat{L}^{(n)}$ appears from the product of n tensors $\delta S(\mathbf{r}_q)\kappa_{j_q}\kappa_{l_{q+1}}$:

$$\hat{L}^{(n)} = \kappa_i\kappa_k\delta S(\mathbf{r})\prod_{p=1}^{n-1} \delta S(\mathbf{r}_p). \tag{24}$$

In what follows we omit the fixed vector $\vec{\kappa}$ in the arguments of the stochastic functions.

The term $\hat{A}^{(n)}$ corresponds to all possible products including one tensor $\delta a_{mn}(\mathbf{r}_s)$ and $n-1$ tensors $\delta S(\mathbf{r}_q)\kappa_{j_q}\kappa_{l_{q+1}}$. When $n = 2$, $\hat{A}^{(2)} = \kappa_i\kappa_{l_1}e_{j_1km}\delta S(\mathbf{r})\delta a_{mn}(\mathbf{r}_1)\kappa_n + \kappa_{j_1}\kappa_k e_{il_1m}\delta S(\mathbf{r}_1)\delta a_{mn}(\mathbf{r})\kappa_n$. If $n > 2$,

$$\hat{A}^{(n)} = e_{il_1m}\frac{\kappa_k}{\kappa_{l_1}}\delta a_{mn}(\mathbf{r})\kappa_n \prod_{p=1}^{n-1} \delta S(\mathbf{r}_p) + e_{j_{n-1}km}\frac{\kappa_i}{\kappa_{j_{n-1}}}\delta S(\mathbf{r})\delta a_{mn}(\mathbf{r}_{n-1})\kappa_n \prod_{p=1}^{n-2} \delta S(\mathbf{r}_p) +$$

$$+\kappa_i\kappa_k \sum_{s=1}^{n-2} e_{j_sl_{s+1}m}\frac{1}{\kappa_{j_s}\kappa_{l_{s+1}}}\delta S(\mathbf{r})\delta a_{mn}(\mathbf{r}_s)\kappa_n \prod_{p=1;p\neq s}^{n-1} \delta S(\mathbf{r}_p). \tag{25}$$

When calculating the tensor $w_{ik}^{(n)}(\mathbf{H})$, we also need to know the tensors $q_{l_sj_s}^{(s)}$ entering the integrand of Eq.(13). Taking account of the expression for the averaged conductivity,

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we see that the contraction $< \sigma_{ik}(\mathbf{H}) > k_i^{(s)} k_k^{(s)} / k_s^2 = (< S > + S_2/H^2)(\vec{\kappa}\vec{k}_s/k_s)^2 + S_1/H^2$. Then, in high magnetic fields in the leading approximation

$$q_{l_s j_s}^{(s)} = \frac{k_{l_s}^{(s)} k_{j_s}^{(s)}}{\left(\vec{\kappa}\vec{k}_s\right)^2 < S >}. \qquad (26.a)$$

Consequently, the leading term of the contraction $q_{l_s j_s}^{(s)} \kappa_{l_s} \kappa_{j_s} = 1/ < S >$, and the contraction

$$X^{(n)} = q_{l_1 j_1}^{(1)} q_{l_2 j_2}^{(2)} \cdots q_{l_{n-1} j_{n-1}}^{(n-1)} \hat{P}^{(n)} = \frac{1}{< S >^{n-1}}. \qquad (26.b)$$

The tensor $\hat{P}^{(n)}$ is defined by Eq.(23).

Let us calculate the contributions of different terms in Eq.(22) to the elements of the tensor $w_{ik}^{(n)}$. We set $w_{ik}^{(n)} = w_{ik}^{(n;L)} + w_{ik}^{(n;A)}/H$. The tensors $w_{ik}^{(n;L)}$, $w_{ik}^{(n;A)}$ are defined by the products $\hat{L}^{(n)}\hat{P}^{(n)}$ and $\hat{A}^{(n)}\hat{P}^{(n)}$, respectively.

We start with calculation of $w_{ik}^{(n;L)}$. Substituting the product $\hat{L}^{(n)}\hat{P}^{(n)}$ in Eq.(13), we see that only the contraction $X^{(n)}$ enters the integrand in the right-hand side of this equation. Then

$$w_{ik}^{(n;L)} = \frac{(-1)^n \kappa_i \kappa_k}{< S >^{n-1}(2\pi)^{3(n-1)}} \int \cdots \int d^3 k_1 ... d^3 k_{n-1} e^{i\mathbf{k}_1 \mathbf{r}} \int \cdots \int d^3 r_1 ... d^3 r_{n-1} \times$$

$$e^{-i\mathbf{r}_1(\mathbf{k}_1 - \mathbf{k}_2)} e^{-i\mathbf{r}_2(\mathbf{k}_2 - \mathbf{k}_3)} ... e^{-i\mathbf{r}_{n-1}\mathbf{k}_{n-1}} < \delta S(\mathbf{r}) \prod_{p=1}^{n-1} \delta S(\mathbf{r}_p) > . \qquad (27.a)$$

After the integration over all the vectors $\mathbf{k}_j$ ($j = 1, 2, ..., n-1$), the δ-functions $\delta(\mathbf{r}_j - \mathbf{r}_{j-1})$ appear in the integrand. These δ-functions allow us to perform integration over position vectors $\mathbf{r}_j$ too. As a result we obtain

$$w_{ik}^{(n;L)} = w_{||}^{(n)} \kappa_i \kappa_k, \quad w_{||}^{(n)} = (-1)^n \frac{< (\delta S)^n >}{< S >^{n-1}}, \qquad (27.b)$$

where $< (\delta S)^n >$ stands for the one-point average $< (\delta S(\mathbf{r}))^n >$.

Before presenting the results for $w_{ik}^{(n;A)}$, let us briefly discuss some statistical properties of polycrystals we need when calculating these tensors. For details see Refs. [11, 16].

The only property of the polycrystalline medium that affects the ensemble averages is the rotations of the crystallographic axes of the grains. Then ensemble average becomes the average over all possible rotations of the crystallites. If in the ensemble the rotations of different grains are statistically independent, when calculating the two-point average $< a(\mathbf{r})b(\mathbf{r}_1) >$ of random physical quantities $a(\mathbf{r})$ and $b(\mathbf{r})$, there are two cases to consider: 1)$\mathbf{r}$ and $\mathbf{r}_1$ are in the same grain, and 2) $\mathbf{r}$ and $\mathbf{r}_1$ are in different grains. We denote by $W_2([\mathbf{r}, \mathbf{r}_1])$ the probability of the case 1. Then $1 - W_2$ is the probability of the case 2. Evidently, $W_2([\mathbf{r}, \mathbf{r}_1]) = 1$, when $\mathbf{r} = \mathbf{r}_1$. Since in the case 2 the two-point average $< a(\mathbf{r})b(\mathbf{r}_1) > = < a(\mathbf{r}) >< b(\mathbf{r}) >$, we have

$$< a(\mathbf{r})b(\mathbf{r}_1) > = < ab > W_2 + (1 - W_2) < a >< b >, \quad < ab > = < a(\mathbf{r})b(\mathbf{r}) > . \qquad (28.a)$$

If $a(\mathbf{r})$ and $b(\mathbf{r})$ are zero-mean quantities ($< a(\mathbf{r}) > = < b(\mathbf{r}) > = 0$), the second term in Eq.(29.a) vanish.

When calculating the three-point average $< a(\mathbf{r})b(\mathbf{r}_1)c(\mathbf{r}_2) >$, let us denote by $W_3([\mathbf{r}_a, \mathbf{r}_b], \mathbf{r}_c)$ the joint conditional probability for the vectors $\mathbf{r}_a$ and $\mathbf{r}_b$ to get in the same grain, and, simultaneously, for the vector $\mathbf{r}_c$ to get in some other grain. The probability $W_3([\mathbf{r}_a, \mathbf{r}_b], \mathbf{r}_c)$ excludes the possibility for all the three vectors to be in the same grain. Next, by $W_3([\mathbf{r}_a, \mathbf{r}_b, \mathbf{r}_c])$ we denote the probability for all the three vectors to get in the same grain. Then

$$< a(\mathbf{r})b(\mathbf{r}_1)c(\mathbf{r}_2) >=< a >< bc > W_3([\mathbf{r}_1, \mathbf{r}_2], \mathbf{r})+ < b >< ac > W_3([\mathbf{r}, \mathbf{r}_2], \mathbf{r}_1)+$$

$$< c >< ab > W_3([\mathbf{r}, \mathbf{r}_1], \mathbf{r}_2)+ < abc > W_3([\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2]); \quad < abc >=< a(\mathbf{r})b(\mathbf{r})c(\mathbf{r}) > .$$
$$(28.b)$$

If $< a(\mathbf{r}) >=< b(\mathbf{r}) >=< c(\mathbf{r}) >= 0$, only the last term in the right-hand side of Eq.(28.b) survives.

When calculating $w_{ik}^{(n;A)}$, let us first examine the tensor $w_{ik}^{(2;A)}$. According to Eq.(13), to calculate $w_{ik}^{(2;A)}$ we need to know the average $< \hat{A}^{(2)} \hat{P}^{(2)} >= \kappa_i \kappa_{j_1} e_{j_1 km} < \delta S(\mathbf{r}) \delta a_{mn}(\mathbf{r}_1) \kappa_n > + \kappa_{l_1} \kappa_k e_{il_1 m} < \delta S(\mathbf{r}_1) \delta a_{mn}(\mathbf{r}) \kappa_n >$. Since $< \delta S(\mathbf{r}) >= 0$ and $< \delta a_{mn}(\mathbf{r}) >= 0$, from Eq.(29.a) it follows that the two-point average $< \delta S(\mathbf{r}) \delta a_{mn}(\mathbf{r}_1) \kappa_n >=< \delta S \delta a_{mn} \kappa_n > W_2([\mathbf{r}, \mathbf{r}_1])$. Since $< \delta S \delta a_{mn} \kappa_n >$ is an isotropic vector, whose components depend on the fixed unit vector $\vec{\kappa}$, it is necessary that $< \delta S \delta a_{mn} \kappa_n >=< \delta S \delta a_{pq} \kappa_p \kappa_q > \kappa_m$. However, according to Eq.(21.b) the contraction $\delta a_{mn} \kappa_m \kappa_n = 0$. Thus, the average $< \hat{A}^{(2)} \hat{P}^{(2)} >= 0$. With regard to Eq.(13) this means that the term $w_{ik}^{(2;A)}$ vanishes.

The same argumentation is valid when calculating the terms $w_{ik}^{(n;A)}$ for an arbitrary $n > 2$. Taking into account the way of calculation of the term $w_{ik}^{(n;L)}$, it is clear that in this case we are faced with calculation of the averages $< (\delta S(\mathbf{r}))^{n-1} \delta a_{mn}(\mathbf{r}_1) \kappa_n >$ and $< \delta S(\mathbf{r}_1))(\delta S(\mathbf{r}))^{n-2} \delta a_{mn}(\mathbf{r}) \kappa_n >$. Because of the equality (21.b), these averages vanish. Thus, for all $n \geq 2$ the tensors $w_{ik}^{(n;A)} = 0$. This also means, that the tensor W_{ik} defined by Eq.(14) does not contain the term of the order of $1/H$.

According to Eq.(27.b) for all n the values of $w_{||}^{(n)}$ do not depend on the statistical properties of the medium. This unusual situation is because of, first, the rather simple tensor form of the fluctuating part of the leading longitudinal element of LCT (according to Eq.(21.b) it is $\delta S(\vec{\kappa}, \vec{r}) \kappa_i \kappa_k$), and, second, the asymptotic form of the tensor $q_{l_s j_s}^{(s)}$ in the limit $H \to \infty$ (see Eq.(26.a)). The first point defines the specific form of the product of n fluctuating parts of LCT (see Eq.(22)). The second point makes it possible to cancel out the common factors $(\vec{\kappa} \vec{k}_s)^2$ in the numerators and denominators of the integrands of Eq.(27.a). This is the reason why no correlators of physical quantities, describing the fluctuating part of LCT in different polycrystalline grains enter the integrands. The absence of the many-point correlators in the final expressions for $w_{||}^{(n)}$ allows us to sum the series and define $W_{||} = \sum_{n=2}^{\infty} w_{||}^{(n)}$. The result is

$$W_{||} =< S > \left\{ < S > \left\langle \frac{1}{S} \right\rangle - 1 \right\}. \tag{29}$$

With respect to the laboratory coordinate system $W_{||} \delta_{i3} \delta_{k3}$ defines the longitudinal part of the tensor W_{ik}.

According to the basic formula (16), to calculate the logitudinal element ECT we also need to define the tensor V_{ik}^{-1}. With regard to the definition (14), to calculate the tensor V_{ik} we need to know tensor I_{ik} (see Eq.(10.b)). This isotropic second rank tensor can be

calculated with the aid of Eq.(18). We set $I_{ik} = I_0 \delta_{ik} + I_1 \kappa_i \kappa_k$. By direct calculation it is easy to see that although each of the coefficients I_0 and I_1 is of the order of H, the sum $I_0 + I_1 = \approx 1/ < S >$. Then with respect to the laboratory coordinate system omitting the terms of the order of $1/H^2$ we have

$$V_{ik} = \delta_{ik} + W_{||}(I_0 + I_1)]\delta_{i3}\delta_{k3}. \tag{30}$$

When with the aid of Eq.(36) we calculate the reciprocal tensor V_{ik}^{-1}, our result for ECT written with respect to the laboratory axes up to the terms of the order of $1/H$, is

$$\sigma_{ik}^{ef}(\mathbf{H})|_{(r_c/l)\ll 1} = \sigma_{||}^{ef}\delta_{i3}\delta_{k3} + \frac{1}{HR_\infty}e_{ik3}, \quad \sigma_{||}^{ef} = \frac{1}{< 1/S >}. \tag{31}$$

The result for $\sigma_{||}^{ef}$ is rather obvious. Indeed, if we calculate ECT up to the terms independent of the small parameter $1/H$ only, with respect to laboratory axes LCT is $\sigma_{ik}(\mathbf{H}, \mathbf{r}) = S(\vec{\kappa}, \vec{r})\delta_{i3}\delta_{k3}$. We can describe such a medium as a set of plane-parallel layers perpendicular to the direction of the magnetic field $\mathbf{H}$. It is known (see, e.g., [1]) that in such a medium the value of the effective conductivity is equal to the inverse the averaged resistivity. This is just our answer for $\sigma_{||}^{ef}$.

From Eq.(31) it follows that the value of the effective Hall conductivity is the same as in the single crystal. Thus, the fluctuation corrections do not manifest themselves in the effective Hall conductivity. From our point of view this is reasonable, since for all the grains the Hall conductivity written with respect to the laboratory coordinate system is the same, and it is equal to the Hall conductivity of single crystal metal.

As an example, let us write down the effective longitudinal conductivity for polycrystals of metals with the Fermi surface that is a uniaxial ellipsoid. Such a surface is the simplest example of a closed nonspherical Fermi surface. With respect to the crystallographic axes, the equation of the Fermi surface is

$$\varepsilon_F = \frac{1}{2m_\perp}p_\perp^2 + \frac{1}{2m_z}p_z^2. \tag{32}$$

As a parameter of the anisotropy of this Fermi surface we use

$$\nu = \frac{m_\perp}{m_z} - 1. \tag{33}$$

Evidently, $-1 < \nu < \infty$. We suppose that electrons are the charge carriers, so that $n_h = 0$ and $n_e = n$.

In the framework of the relaxation time approximation the general expression for the conductivity of single crystal metals with an ellipsoidal Fermi surface in an arbitrary magnetic field was obtained in [17]. From the results of [17] it follows that in high magnetic fields the longitudinal local conductivity is

$$S(\vec{\kappa}) = \frac{\sigma_z}{1 + \nu(1 - \cos^2\theta)}; \quad \sigma_z = \frac{ne^2\tau}{m_z},$$

where θ is the angle between the principle axes z of the ellipsoid (32) and the direction of the magnetic field $\vec{\kappa}$.

From Eq.(31) it follows that in this case the effective longitudinal conductivity is

$$\sigma_{||}^{ef} = \frac{3\sigma_z(1+\nu)}{3+2\nu}. \tag{34}$$

The perturbation series for $\sigma_{||}^{ef}$ converges when $\delta S/<S><1$. Direct calculations show that this inequality is fulfilled at least if $\nu < 1.7$.

Concluding this Section some remarks on calculation of the transverse part of the conductivity tensor has to be done. It can appear that if in Eq.(22) we write the product of stochastic tensors $\Delta_{il_1}(\mathbf{H},\mathbf{r})\Delta_{j_1l_2}(\mathbf{H},\mathbf{r}_1)...\Delta_{j_{n-1}k}(\mathbf{H},\mathbf{r}_{n-1})$ up to the terms of the order of $1/H^2$, we can calculate the effective transverse conductivity too. Indeed, the calculation similar to the one presented above, show that this terms define the contribution $- < \delta a_{mp}\delta a_{mq}\kappa_p\kappa_q > /2 < S >$ to the transverse part of the tensor W_{ik} in the basic Eq.(16). We see that this term is of the second order in the stochastic part of the tensor a_{mp}. However, as it follows from the results of Yu.A.Dreizin and A.M.Dykhne (see [15]) when calculating this term we cannot restrict ourselves with the leading approximation for the tensor $q_{l_sj_s}^{(s)}$ only. The poles entering the elements of this tensor provide the contribution to the transverse conductivity following from the terms of the order higher than $1/H^2$ in the product $\Delta_{il_1}(\mathbf{H},\mathbf{r})\Delta_{j_1l_2}(\mathbf{H},\mathbf{r}_1)...\Delta_{j_{n-1}k}(\mathbf{H},\mathbf{r}_{n-1})$. Of course, is the anisotropy is very small, these terms can be omitted, but this resoning indicates, that most likely no exact solution can be obtained for the transverse effective conductivity in high magnetic fields. In any case further analysis is necessary to obtain the correct result.

4   Conclusions

Concluding this paper some remarks are to the point. First, when galvanomagnetic phenomena are investigated experimentally, the object of investigation is the resistivity tensor. If we know the leading term of the elements of ECT up to the terms of the order of $1/H$ exactly, we can use Eqs.(31) when calculating the asymptotic of the logitudinal and the Hall effective resistivities $(\rho_{ik}^{ef}(\mathbf{H})\sigma_{ik}^{ef}(\mathbf{H}) = \delta_{ik})$. Then

$$\rho_{||}^{ef} = < 1/S >, \quad \rho_{12}^{ef} = R_\infty H. \tag{35}$$

We see that again the effective Hall element of $\rho_{ik}^{ef}(\mathbf{H})$ is the same as in the single crystal.

Second, usually, one thinks about polycrystals as of isotropic metals, that is as of metals with spherical Fermi surface. However, our results show that this idea is not justified. Indeed, for isotropic metals in high magnetic fields the longitudinal element of the conductivity tensor is $\sigma_{||}^{sph} = \sigma_0$, where σ_0 is the conductivity when $H = 0$. Let us verify this equality for polycrystal with ellipsoidal Fermi surface. From the results of [11] it follows that if $\nu < 1$, with high accuracy we can estimate the effective value of σ_0 as $< \sigma(0) >$. Then for the case of ellipsoidal Fermi surface we have

$$\frac{\sigma_{||}^{ef}}{<\sigma(0)>} = \frac{9(1+\nu)}{(3+2\nu)(3+\nu)}.$$

Evidently, this ratio is not equal to one.

Finally, when comparing the calculated ECT with experimental results, one must have in mind that in our calculations only the inhomogeneity due to different orientations of

polycrystalline grains was taken into account. Of course, in real polycrystals there are other sources of inhomogeneity too. For example, we do not take into account the real structure of the boundaries of the grains. This simplification is justified when the grains are sufficiently large and the properties of the grain boundaries do not affect the result significantly.

References

[1] A.G.Fokin, Physics - Uspekhi 39 (1996) 1009.

[2] A.M.Dykhne, JETP 59 (1970) 110.

[3] A.M.Dykhne, JETP 59 (1970) 641.

[4] S.A.Bulgadaev, Phys. Lett. A, 313 (2003) 144.

[5] A.M.Dykhne, I.M.Kaganova, Physica A, 241, (1997), 154.

[6] A.M.Dykhne, I.M.Kaganova, Phys. Rep. 288, (1997), 263.

[7] I.M.Kaganova, M.I.Kaganov, Phys. Rev. B, 63 (2001), 054202.

[8] A.M.Dykhne, I.M.Kaganova, Opt. Commun. 206 (2002), 39.

[9] Yu.A.Dreizin, A.M.Dykhne, JETP, 84, (1983), 1756.

[10] I.M.Lifshitz, L.N.Rosenzweig, Zh. Eksp. Teor. Fiz. (JETP), 16, (1946), 967.

[11] I.M.Kaganova, Phys. Lett. A, 312 (2003) 108.

[12] I.M.Lifshitz, M.Ya.Asbel', M.I.Kaganov, *Electron Theory of Metals* (New York, Consultants Bureau, 1973); E.M.Lifshitz and P.L.Pitaevskii, *Physical Kinetics* (Pergamon, Oxford, 1981).

[13] A.P.Cracknell, K.C.Wong, *The Fermi Surface. Its concept, determination, and use in the physics of metals* (Clarendon Press, Oxford, 1973).

[14] I.M.Lifshitz, V.G.Peschanskii, JETP, 35, (1958), 1251. (1946), 967.

[15] Yu.A.Dreizin, A.M.Dykhne, JETP, 63, (1972), 242.

[16] I.M.Kaganova, A.A.Maradudin, Phys. Scr. T44, (1992), 104.

[17] F.G.Bass, M.I.Kaganov and V.V.Slyezov, Fizika metallov I metallovedenie (in Russian), 5, (1957), 406